The normal distribution

13.1 Kick off with CAS
13.2 The normal distribution
13.3 Calculating probabilities and the standard normal distribution
13.4 The inverse normal distribution
13.5 Mixed probability application problems
13.6 Review
The normal distribution

The normal distribution is a type of continuous probability distribution. It has a distinct bell-shaped curve and is given by the equation

\[ f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x - \mu}{\sigma} \right)^2} \]

where \( \mu \) is the mean of the data and \( \sigma \) is the standard deviation.

A normal distribution curve can be sketched in CAS via the inbuilt probability programs or by using the equation of the function.

Exploring the effect of \( \mu \) on the normal distribution

1. On the same set of axes, sketch the following normal distributions.
   a. \( \mu = 50, \sigma = 10 \)
   b. \( \mu = 20, \sigma = 10 \)
   c. \( \mu = 75, \sigma = 10 \)

2. What effect does changing the value of the mean have on the shape of the normal distribution curve?

Exploring the effect of \( \sigma \) on the normal distribution

3. On the same set of axes, sketch the following normal distributions.
   a. \( \mu = 50, \sigma = 5 \)
   b. \( \mu = 50, \sigma = 10 \)
   c. \( \mu = 50, \sigma = 15 \)

4. What effect does changing the value of the standard deviation have on the shape of the normal curve?

5. Given that the normal distribution curve is a probability density function, can you explain the effect of \( \sigma \) on the graph?

Please refer to the Resources tab in the Prelims section of your eBookPLUS for a comprehensive step-by-step guide on how to use your CAS technology.
The normal distribution is arguably the most important distribution in statistics. It is characterised by the well-known bell-shaped curve, which is symmetrical about the mean (as well as the median and mode). Continuous random variables such as height, weight, time and other naturally occurring phenomena are frequently analysed with normal distribution calculations.

Normal distributions may vary depending on their means and standard deviations. The following diagram shows three different normal distributions.

Graph 1 has mean of $-1$ and a standard deviation of $0.5$.

Graph 2 has a mean of $0$ and a standard deviation of $1$.

Graph 3 has a mean of $3$ and a standard deviation of $0.25$.

The probability density function for the normal distribution is given by

$$f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x - \mu}{\sigma}\right)^2}$$

where the parameters $\mu$ and $\sigma$ are the mean and standard deviation of the distribution respectively.

We say that

$$X \sim N(\mu, \sigma^2)$$

meaning $X$ is distributed normally with the mean and variance specified.

There are five important characteristics of the normal distribution.

1. A normal distribution is symmetrical about the mean.
2. The mean, median and mode are equal.
3. The area under the curve is equal to 1. That is, $\int_{-\infty}^{\infty} f(x) \, dx = 1$.
4. The majority of the values feature around the centre of the curve with fewer values at the tails of the curve.
5. Normal distributions are defined by two parameters – the mean, $\mu$, and the standard deviation, $\sigma$.

As the mean and standard deviation can vary, and the area under the graph must be constant and equal to 1, in effect, changing the mean and the standard deviation transforms the normal curve.
Changing the standard deviation affects the normal curve twofold. The transformed curve will display:

- dilation by a factor $\frac{1}{\sigma}$ parallel to the $y$-axis
- dilation by a factor $\sigma$ parallel to the $x$-axis.

Changing the mean has the effect of a translation parallel to the $x$-axis.

The importance of the normal distribution stems from the fact that the distributions of many naturally occurring phenomena can be approximated by this distribution. The pattern was first noticed by astronomers in the seventeenth century. Galileo realised that errors in astronomical observation formed a symmetrical curve and that small errors occurred more frequently than large errors. However, it was not until the nineteenth century that the formula to describe this distribution was developed, by the German mathematician Carl Friedrich Gauss.

In summary, the normal probability density function has the following characteristics:

- $f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{x - \mu}{\sigma} \right)^2}$, $x \in \mathbb{R}$
- The distribution is symmetrical about the mean.
- $\mu = \text{median} = \text{mode}$
- The maximum value is $\frac{1}{\sigma \sqrt{2\pi}}$ when $x = \mu$.
- The curve continues infinitely in both directions.
- $\int_{-\infty}^{\infty} f(x) dx = 1$

Important intervals and their probabilities

Often we are required to find the proportion of a population for a given interval. Using the property that the symmetry of the normal distribution is about the mean, we are able to predict with certainty the following facts.

- Approximately 68% of the population will fall within 1 standard deviation of the mean:
  \[ \Pr(\mu - \sigma < X < \mu + \sigma) \approx 0.68. \]
- Approximately 95% of the population will fall within 2 standard deviations of the mean:
  \[ \Pr(\mu - 2\sigma < X < \mu + 2\sigma) \approx 0.95. \]

We say that a randomly chosen member of the population will most probably be or is highly likely to be within 2 standard deviations of the mean.
Approximately 99.7% of the population will fall within 3 standard deviations of the mean:

\[ \Pr(\mu - 3\sigma < X < \mu + 3\sigma) \approx 0.997. \]

We say that a randomly chosen member of the population will almost certainly be within 3 standard deviations of the mean. This is shown on the following graphs.

A more comprehensive breakdown of the proportion of the population for each standard deviation is shown on the graph below.

**WORKED EXAMPLE 1**

The probability density function for a normal distribution is given by

\[ f(x) = \frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}(2(x-\mu))^2}, \quad x \in \mathbb{R}. \]

a. State the mean and standard deviation of the distribution.

b. Sketch the graph of the function.
THINK

a Use \( f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x - \mu}{\sigma})^2} \) to

determine \( \mu \) and \( \sigma \).

b Sketch the graph with a mean of 1 and a

standard deviation of 0.5. The x-axis needs to

be scaled with markings at \( \mu \), \( \mu \pm \sigma \), \( \mu \pm 2\sigma \)

and \( \mu \pm 3\sigma \). The peak of the graph must also

be labelled with its coordinates.

WRITE/DRAW

a \( f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x - \mu}{\sigma})^2} \)

\[ = \frac{2}{\sqrt{2\pi}} e^{-\frac{1}{2}(2(x-1))^2} \]

\[ \frac{1}{\sigma} = 2, \text{ so } \sigma = \frac{1}{2} \text{ and } \mu = 1. \]

b \[ f(x) = \frac{2}{\sqrt{2\pi}}, \quad \left(1, \frac{2}{\sqrt{2\pi}}\right) \]

WORKED EXAMPLE

The heights of the women in a particular town are normally distributed with

a mean of 165 centimetres and a standard deviation of 9 centimetres.

a What is the approximate probability that a woman chosen at random has a

height which is between 156 cm and 174 cm?

b What is the approximate probability that a woman chosen at random is
taller than 174 cm?

c What approximate percentage of the women in this particular town are
shorter than 147 cm?

THINK

a Determine how many standard

deviations from the mean the

156–174 cm range is.

b Use the fact that

\( P(156 \leq X \leq 174) \approx 0.68 \) to
calculate the required probability.

Sketch a graph to help.

WRITE/DRAW

a Let \( X \) be the height of women in this particular town.

\[ \mu + \sigma = 165 + 9 \]

\[ = 174 \]

\[ \mu - \sigma = 165 - 9 \]

\[ = 156 \]

Since the range is one standard deviation from the mean,

\( \Pr(156 \leq X \leq 174) \approx 0.68. \)

b

\[ 0.68 \]

\[ 156 \quad 165 \quad 174 \]

\( x \)
The normal distribution

The probability density function of a normal distribution is given by

\[ f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x - \mu)^2} \]

1. **a** State the mean and the standard deviation of the distribution.

2. **b** Sketch the graph of the probability function.

3. **c** Determine how many standard deviations 147 cm is from the mean.

**c** \[ \mu - \sigma = 165 - 9 = 156 \]

\[ \mu - 2\sigma = 165 - 2 \times 9 = 147 \]

147 cm is 2 standard deviations from the mean. The corresponding upper value is 183 (165 + 2 \times 9).

\[ \Pr(147 \leq X \leq 183) \approx 0.95 \]

**EXERCISE 13.2**

**1. WE1** The probability density function of a normal distribution is given by

\[ f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x - \mu)^2} \]

- **a** State the mean and the standard deviation of the distribution.
- **b** Sketch the graph of the probability function.

2. A normal distribution has a probability density function of

\[ f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x - 3)^2} \]
a Using CAS technology, verify that \[ \int_{-\infty}^{\infty} f(x) \, dx = 1. \]

b State \( \mu \) and \( \sigma \).

c Sketch the graph of the probability function.

3 The results of a Mathematical Methods test are normally distributed with a mean of 72 and a standard deviation of 8.

a What is the approximate probability that a student who sat the test has a score which is greater than 88?

b What approximate proportion of the students who sat the test had a score which was less than 48?

c What approximate percentage of the students who sat the test scored less than 80?

4 The length of pregnancy for a human is normally distributed with a mean of 275 days and a standard deviation of 14 days. A mother gave birth after less than 233 days. What is the approximate probability of this happening for the general population?

5 Consider the normal probability density function
\[ f(x) = \frac{1}{4 \sqrt{2\pi}} e^{-\frac{1}{2}\left(x + \frac{2}{4}\right)^2}, \quad x \in \mathbb{R}. \]

a Using CAS technology, verify that \( \int_{-\infty}^{\infty} f(x) \, dx = 1. \)

b State \( \mu \).

6 A normal probability density function is defined by
\[ f(x) = \frac{10}{3 \sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{10(x - 1)}{3}\right)^2}, \quad x \in \mathbb{R}. \]

a Find the values of \( \mu \) and \( \sigma \).

b State what effect the mean and standard deviation have on the graph of the normal distribution.

c Sketch the graph of the function, \( f \).

7 A normal probability density function is given by
\[ f(x) = \frac{1}{10 \sqrt{2\pi}} e^{-\frac{1}{2}\left(x + \frac{4}{10}\right)^2}, \quad x \in \mathbb{R}. \]

a Find the values of \( \mu \) and \( \sigma \).

b State what effect the mean and standard deviation have on the graph of the normal distribution.

c Determine:
   i \( \text{Var}(X) \) \qquad \text{ii} \ E(X^2).

d Verify that this is a probability density function.

8 \[ f(x) = \frac{5}{2 \sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{5(x - 2)}{2}\right)^2}, \quad x \in \mathbb{R} \] defines a normal probability density function.

a Find the values of \( \mu \) and \( \sigma \).

b Calculate \( E(X^2) \).

c Determine:
   i \( E(5X) \) \qquad \text{ii} \ E(5X^2).
Scores on a commonly used IQ test are known to be normally distributed with a mean of 120 and a standard deviation of 20.

**a** Determine:
- $\mu \pm \sigma$
- $\mu \pm 2\sigma$
- $\mu \pm 3\sigma$.

**b** Find:
- $\Pr(X < 80)$
- $\Pr(X > 180)$.

The results of a Year 12 Biology examination are known to be normally distributed with a mean of 70 and a standard deviation of 6. What approximate percentage of students sitting for this examination can be expected to achieve a score that is greater than 88?

A continuous random variable, $X$, is known to be normally distributed with a mean of 15 and a standard deviation of 5. Find the range between which approximately:
- 68% of the values lie
- 95% of the values lie
- 99.7% of the values lie.

A normal probability density function, $X$, has a mean of 24 and a standard deviation of 7. Find the approximate values for:
- $\Pr(X < 31)$
- $\Pr(10 < X < 31)$
- $\Pr(X > 10 | X < 31)$.

The number of pears harvested from each tree in a large orchard is normally distributed with a mean of 230 and a standard deviation of 25. Find the approximate probability that the number of pears harvested from a randomly selected tree is:
- less than 280
- between 180 and 280
- is greater than 180, given that less than 280 pears were harvested.

The annual rainfall in a particular area of Australia, $X$ mm, is known to be normally distributed with a mean of 305 mm and a standard deviation of 50 mm.

- Calculate the approximate value of $\Pr(205 < X < 355)$.
- Find $k$ such that $\Pr(X < k) \approx 0.025$.
- Find $h$ such that $\Pr(X < h) \approx 0.0015$.

A normally distributed probability density function is given by

$$f(x) = \frac{5}{\sqrt{2\pi}} e^{-\frac{1}{2}(5(x - 1))^2}, \ x \in \mathbb{R}.$$ 

- Calculate $\text{Var}(X)$, giving your answer correct to 2 decimal places.
- Calculate $\text{E}(X^2)$, giving your answer correct to 2 decimal places.
- Find:
  - $\text{E}(2X + 3)$
  - $\text{E}((X + 1)(2X - 3))$.

A continuous random variable, $X$, is normally distributed with a mean of 72.5 and a standard deviation of 8.4. Find the approximate values for:
- $\Pr(64.1 < X < 89.3)$
- $\Pr(X < 55.7)$
- $\Pr(X > 47.3 | X < 55.7)$
- $m$ such that $\Pr(X > m) \approx 0.16$. 

---

**504 MATHS QUEST 12 MATHEMATICAL METHODS VCE Units 3 and 4**
Calculating probabilities and the standard normal distribution

The standard normal distribution

Suppose we are comparing the results of two students on two similar IQ tests. Michelle obtained 92 on one IQ test, for which the results were known to be normally distributed with a mean of 80 and a standard deviation of 6. Samara obtained 88 on a similar IQ test, for which the results were known to be normally distributed with a mean of 78 and a standard deviation of 10. Which student was the most successful?

This question is very difficult to answer unless we have some common ground for a comparison. This can be achieved by using a transformed or standardised form of the normal distribution called the standard normal distribution. The variable in a standard normal distribution is always denoted by \( Z \), so that it is immediately understood that we are dealing with the standard normal distribution. The standard normal distribution always has a mean of 0 and a standard deviation of 1, so that \( Z \) indicates how many standard deviations the corresponding \( X \)-value is from the mean. To find the value of \( Z \), we find the difference between the \( x \)-value and the mean, \( x - \mu \). To find how many standard deviations this equals, we divide by the standard deviation, \( \sigma \).

\[
Z = \frac{x - \mu}{\sigma}
\]

Therefore, if \( Z = \frac{x - \mu}{\sigma} \), \( \mu = 0 \) and \( \sigma = 1 \), the probability density function is given by

\[
f(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}, \quad z \in \mathbb{R}.
\]

Remember that \( \mu \pm 3\sigma \) encompasses approximately 99.7% of the data, so for the standard normal curve, these figures are \( 0 \pm 3 \times 1 = 0 \pm 3 \).

Therefore, approximately 99.7% of the data lies between \(-3 \) and 3.

For the standard normal distribution, we say \( Z \sim N(0, 1) \).

Let us return to the comparison between Michelle and Samara.

For Michelle: \( X \sim N(80, 6^2) \), \( z = \frac{x - \mu}{\sigma} = \frac{92 - 80}{6} = \frac{12}{6} = 2 \)

For Samara: \( X \sim N(78, 10^2) \), \( z = \frac{x - \mu}{\sigma} = \frac{88 - 78}{10} = \frac{10}{10} = 1 \)

Michelle’s mark lies within 2 standard deviations of the mean, so it lies in the top 2.5%, whereas Samara’s mark is 1 standard deviation from the mean, so it is in the top 16%. Hence, Michelle performed better than Samara.
Obviously, not all data values will lie exactly 1, 2 or 3 standard deviations from the mean. In these cases technology such as a CAS calculator is needed to calculate the required probability. CAS can be used to calculate probabilities associated with the normal distribution for any value of $\mu$ and $\sigma$.

**WORKED EXAMPLE 3**

a. Calculate the values of the following probabilities, correct to 4 decimal places.
   i. $\Pr(Z < 2.5)$
   ii. $\Pr(-1.25 \leq Z \leq 1.25)$

b. $X$ is a normally distributed random variable such that $X \sim N(25, 3^2)$.
   i. Calculate $\Pr(X > 27)$ correct to 4 decimal places.
   ii. Convert $X$ to a standard normal variable, $Z$.

**THINK**

a. i. Sketch a graph to help understand the problem.

b. i. Sketch a graph to help understand the problem.

**WRITE/DRAW**

a. i. $f(z)$

   ![Graph of f(z)]

   Pr($z < 2.5$) = 0.9938

ii. Sketch a graph to help understand the problem.

b. i. Sketch a graph to help understand the problem.

ii. Sketch a graph to help understand the problem.

Pr$(-1.25 < Z < 1.25)$ = 0.7887

**WORKED EXAMPLE 3**

3 Sketch a graph to help understand the problem.

3 Use CAS to find the probability.
   The upper limit is 2.5 and the lower limit is $-\infty$.
   The mean is 0 and the standard deviation is 1.

ii. Sketch a graph to help understand the problem.

3 Use CAS to find the probability.
   The upper limit is 1.25 and the lower limit is $-1.25$.

b. i. Sketch a graph to help understand the problem.

3 Sketch a graph to help understand the problem.

3 Use CAS to find the probability.
   The upper limit is 2.5 and the lower limit is $-\infty$.
   The mean is 0 and the standard deviation is 1.

ii. Sketch a graph to help understand the problem.

3 Use CAS to find the probability.
   The upper limit is 1.25 and the lower limit is $-1.25$.

b. i. Sketch a graph to help understand the problem.

3 Sketch a graph to help understand the problem.

3 Use CAS to find the probability.
   The upper limit is 2.5 and the lower limit is $-\infty$.
   The mean is 0 and the standard deviation is 1.

ii. Sketch a graph to help understand the problem.
Calculating probabilities and the standard normal distribution

1. Calculate the values of the following probabilities correct to 4 decimal places.
   - i. \( \Pr(Z < 1.2) \)
   - ii. \( \Pr(-2.1 < Z < 0.8) \)

2. \( X \) is a normally distributed random variable such that \( X \sim N(45, 6^2) \).
   - i. Calculate \( \Pr(X > 37) \) correct to 4 decimal places.
   - ii. Convert \( X \) to a standard normal variable, \( Z \).

3. For a particular type of laptop computer, the length of time, \( X \) hours, between charges of the battery is normally distributed such that \( X \sim N(50, 15^2) \).
   - Find \( \Pr(50 < X < 70) \).

4. If \( Z \sim N(0, 1) \), find:
   - a. \( \Pr(Z \leq 2) \)
   - b. \( \Pr(Z \leq -2) \)
   - c. \( \Pr(-2 < Z \leq 2) \)
   - d. \( \Pr(Z > 1.95) \cup \Pr(Z < -1.95) \).

5. Convert the variable in the following expressions to a standard normal variable, \( Z \), and use it to write an equivalent expression. Use your calculator to evaluate each probability.
   - a. \( \Pr(X < 61), X \sim N(65, 9) \)
   - b. \( \Pr(X \geq 110), X \sim N(98, 225) \)
   - c. \( \Pr(-2 < X \leq 5), X \sim N(2, 9) \).

5. A radar gun is used to measure the speeds of cars on a freeway. The speeds are normally distributed with a mean of 98 km/h and a standard deviation of 6 km/h. What is the probability that a car picked at random is travelling at:
   - a. more than 110 km/h
   - b. less than 90 km/h
   - c. a speed between 90 km/h and 110 km/h?
A large number of students took a test in Physics. Their final grades have a mean of 72 and a standard deviation of 12. If the distribution of these grades can be approximated by a normal distribution, what percentage of students, correct to 2 decimal places:

a gained a score of more than 95
b should pass the test if grades greater than or equal to 55 are considered passes?

7 \(X\) is a continuous random variable and is known to be normally distributed.

a If \(\Pr(X < a) = 0.35\) and \(\Pr(X < b) = 0.62\), find:
   i \(\Pr(X > a)\)
   ii \(\Pr(a < X < b)\).

b If \(\Pr(X < c) = 0.27\) and \(\Pr(X < d) = 0.56\), find:
   i \(\Pr(c < X < d)\)
   ii \(\Pr(X > c | X < d)\).

c A random variable, \(X\), is normally distributed with a mean of 20 and a standard deviation of 5.
   i Find \(k\) if \(\Pr(X > 32) = \Pr(Z > k)\).
   ii Find \(k\) if \(\Pr(X < 12) = \Pr(Z > k)\).

Jing Jing scored 85 on the mathematics section of a scholarship examination, the results of which were known to be normally distributed with a mean of 72 and a standard deviation of 9. Rani scored 18 on the mathematics section of a similar examination, the results of which were normally distributed with a mean of 15 and a standard deviation of 4. Assuming that both tests measure the same kind of ability, which student has the higher score?

9 A salmon farm in Tasmania has a very large number of salmon in its ponds.
   It is known that the lengths of the salmon from this farm are normally distributed with a mean of 38 cm and a standard deviation of 2.4 cm. A randomly chosen fish from this farm was measured as 39.5 cm. If salmon with lengths in the top 15% are considered to be gourmet salmon, determine whether this particular fish can be classified as gourmet.

10 The results by Justine in Chemistry, Mathematical Methods and Physics are shown in the table below. The marks, \(X\), the mean, \(\mu\), and standard deviation, \(\sigma\), for each examination are given.

<table>
<thead>
<tr>
<th>Subject</th>
<th>Mark, (X)</th>
<th>Mean, (\mu)</th>
<th>Standard deviation, (\sigma)</th>
<th>Standardised mark, (Z)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chemistry</td>
<td>72</td>
<td>68</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>Maths Methods</td>
<td>75</td>
<td>69</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>Physics</td>
<td>68</td>
<td>61</td>
<td>8</td>
<td></td>
</tr>
</tbody>
</table>

Complete the table by finding Justine’s standardised mark for each subject and use this to determine in which subject she did best when compared to her peers.
11 Teresa has taken her pulse each day for a month after going for a brisk walk. Her pulse rate in beats per minute is known to be normally distributed with a mean of 80 beats per minute and a standard deviation of 5 beats per minute. After her most recent walk she took her pulse rate. What is the probability that her pulse rate was:

a in excess of 85 beats per minute
b equal to or less than 75 beats per minute
c between 78 and 82 beats per minute, given that it was higher than 75 beats per minute?

12 The labels on packets of sugar say the bags have a weight of 1 kg. The actual mean weight of the bags is 1.025 kg in order to minimise the number of bags which may be underweight. If the weight of the bags is normally distributed with a standard deviation of 10 g, find the percentage of bags that would be expected to weigh:

a more than 1.04 kg
b less than 996 g, the legal meaning of underweight?

13 If $Z \sim N(0, 1)$, find:

a $\Pr(Z \geq 2.125) \cup \Pr(Z < -2.125)$
b $\Pr(X < 252.76) \text{ if } \mu = 248.55, \sigma = 24.45 \text{ and } X \text{ is normally distributed}$
c $\Pr(-3.175 \leq Z \leq 1.995)$
d $\Pr(X < 5.725) \text{ if } \mu = 7.286, \sigma = 3.115 \text{ and } X \text{ is normally distributed}.$

14 A continuous random variable, $Z$, has a probability density function defined by $f(z) = 0.025e^{-0.025z}$, $z \geq 0$.

A second continuous random variable, $Y$, is distributed normally with a mean of 25 and a standard deviation of 3. In order to find $k$ such that $\Pr(Z > k) = \Pr(Y < k)$,

$$\int_{k}^{\infty} f(z) \, dz = \int_{-\infty}^{0} g(y) \, dy$$

must be solved. Find the value of $k$.

13.4 The inverse normal distribution

CAS technology provides an easy way to find a $Z$ or $X$ value, given a probability for a normal distribution. Suppose $X$ is normally distributed with a mean of 32 and a standard deviation of 5. We wish to find $\Pr(X \leq a) = 0.72$.

The key information to enter into your calculator is the known probability, that is, the area under the curve. It is essential to input the correct area so that your calculator knows if you are inputting the ‘less than’ area or the ‘greater than’ area.
Quantiles and percentiles

Quantiles and percentiles are terms that enable us to convey information about a distribution. Quantiles refer to the value below which there is a specified probability that a randomly selected value will fall. For example, to find the 0.7 quantile of a standard normal distribution, we find $a$ such that $\Pr(Z < a) = 0.7$.

Percentiles are very similar to quantiles. For the example of $\Pr(Z < a) = 0.7$, we could also be asked to find the 70th percentile for the standard normal distribution.

WORKED EXAMPLE 4

If $X$ is a normally distributed random variable, find:

a $m$ given that $\Pr(X \leq m) = 0.85$, $X \sim N(15.2, 1.5^2)$

b $n$ given that $\Pr(X > n) = 0.37$, $X \sim N(22, 2.75^2)$

c $p$ given that $\Pr(37.6 - p \leq X \leq 37.6 + p) = 0.65$, $X \sim N(37.6, 12^2)$.

THINK

a Use the probability menus on the CAS calculator to find the required $X$ value.

b Use the probability menus on the CAS calculator to find the required $X$ value. 

Note: It may be a requirement to input the ‘less than’ area, so 
$\Pr(X < n) = 1 - 0.37$
$= 0.63$

c Sketch a graph to visualise the problem. Due to symmetry, the probabilities either side of the upper and lower limits can be calculated.

WRITE/DRAW

a $\Pr(X \leq m) = 0.85$, $\mu = 15.2$, $\sigma = 1.5$
$m = 16.7547$

b $\Pr(X > n) = 0.37$, $\mu = 22$, $\sigma = 2.75$
$n = 22.9126$

c

1 Sketch a graph to visualise the problem. Due to symmetry, the probabilities either side of the upper and lower limits can be calculated.

$\Pr(X < 37.6 - p) = \Pr(X > 37.6 + p)$
$= \frac{0.35}{2}$
$= 0.175$

2 Determine $p$ by finding $X$ given that $\Pr(X < 37.6 - p) = 0.175$.

Note: $p$ could also be found by using the upper limit.

$37.6 - p = 26.38$
$p = 37.6 - 26.38$
$p = 11.22$
Finding the mean or standard deviation

If the mean or standard deviation is unknown, then the known probability needs to be linked to the standard normal distribution and the corresponding \( z \)-value calculated via CAS. Once the \( z \)-value has been found, the missing mean or standard deviation can be calculated via the rule \( z = \frac{x - \mu}{\sigma} \).

**WORKED EXAMPLE 5**

a  For the normally distributed variable \( X \), the 0.15 quantile is 1.9227 and the mean is 2.7. Find the standard deviation of the distribution.

b  \( X \) is normally distributed so that the 63rd percentile is 15.896 and the standard deviation is 2.7. Find the mean of \( X \).

**THINK**

a  1 Write the probability statement.

2 Find the corresponding standardised value, \( Z \), by using CAS.

3 Write the standardised formula connecting \( z \) and \( x \).

4 Substitute the appropriate values and solve for \( \sigma \).

b  1 Write the probability statement.

2 Find the corresponding standardised value, \( Z \), by using CAS.

3 Write the standardised formula connecting \( z \) and \( x \).

4 Substitute in the appropriate values and solve for \( \mu \).

**WRITE**

a  The 0.15 quantile is 1.9227.

Pr(\( X < 1.9227 \)) = 0.15

\[ \Pr(Z < z) = 0.15 \]

\[ z = -1.0364 \]

\[ z = \frac{x - \mu}{\sigma} \]

\[ -1.0364 = \frac{1.9227 - 2.7}{\sigma} \]

\[ -1.0364\sigma = -0.7773 \]

\[ \sigma = 0.75 \]

b  The 63rd percentile is 15.896.

Pr(\( X < 15.896 \)) = 0.63

\[ \Pr(Z < z) = 0.63 \]

\[ z = 0.3319 \]

\[ z = \frac{x - \mu}{\sigma} \]

\[ 0.3319 = \frac{15.896 - \mu}{2.7} \]

\[ 0.8960 = 15.896 - \mu \]

\[ \mu = 15 \]

**EXERCISE 13.4**

**PRACTISE**

**The inverse normal distribution**

1  **WE4** Find the value of \( a \), correct to 2 decimal places, if \( X \) is normally distributed and:

a  \( \Pr(X \leq a) = 0.16, X \sim N(41, 6.7^2) \)

b  \( \Pr(X > a) = 0.21, X \sim N(12.5, 2.7^2) \)

c  \( \Pr(15 - a \leq X \leq 15 + a) = 0.32, X \sim N(15, 4^2) \).

2  Find the values of \( m \) and \( n \) if \( X \) is normally distributed and \( \Pr(m \leq X \leq n) = 0.92 \) when \( \mu = 27.3 \) and \( \sigma = 8.2 \). The specified interval is symmetrical about the mean.
3. $X$ is distributed normally with a mean of 112, and the 42nd percentile is 108.87. Find the standard deviation of the distribution, correct to 1 decimal place.

4. $X$ is a normally distributed random variable such that $X \sim N(\mu, 4.45^2)$. If the 0.11 quantile is 32.142, find the value of $\mu$, correct to 1 decimal place.

5. If $Z \sim N(0, 1)$ find the value of $z$ given that:
   a. $\Pr(Z < z) = 0.39$
   b. $\Pr(Z > z) = 0.15$
   c. $\Pr(-z < Z < z) = 0.28$.

6. If $X \sim N(37.5, 8.62^2)$, find $a$ correct to 2 decimal places such that:
   a. $\Pr(X < a) = 0.72$
   b. $\Pr(X > a) = 0.32$
   c. $\Pr(37.5 - a < X < 37.5 + a) = 0.88$.

7. For a standard normal distribution, find:
   a. the 0.57 quantile
   b. the 63rd percentile.

8. If $X$ is distributed normally with $\mu = 43.5$ and $\sigma = 9.7$, find:
   a. the 0.73 quantile
   b. the 24th percentile.

9. $X$ is distributed normally with a standard deviation of 5.67, and $\Pr(X > 20.952) = 0.09$. Find the mean of $X$, giving your answer correct to 2 decimal places.

10. $X$ is distributed normally with a standard deviation of 3.5, and $\Pr(X < 23.96) = 0.28$. Find the mean for $X$, rounded to the nearest whole number.

11. $X \sim N(115, \sigma^2)$ and the 76th percentile is 122.42. Find the value of $\sigma$, giving your answer correct to 1 decimal place.

12. $X$ is distributed normally with $\mu = 41$ and $\Pr(X > 55.9636) = 0.11$. Find $\sigma$, giving your answer correct to 1 decimal place.

13. $X$ is distributed normally and $\Pr(X < 33.711) = 0.36$ while $\Pr(X < 34.10) = 0.42$. Find the mean and the standard deviation of $X$, giving your answer correct to 1 decimal place.

14. $X$ is distributed normally such that $\Pr(X > 18.35) = 0.31$ and the 45th percentile is 15.09. Find $\mu$ and $\sigma$ for $X$, giving your answer correct to 1 decimal place.

15. $\Pr(a < X < b) = 0.52$ and the specified interval is symmetrical about the mean. If $X$ is normally distributed with a mean of 42.5 and a standard deviation of 10.3, find $\Pr(X > a | X < b)$.

16. $X$ is distributed normally such that $\Pr(X < 39.9161) = 0.5789$ and $\Pr(X > 38.2491) = 0.4799$. Find the mean and the standard deviation of $X$, giving your answers correct to 2 decimal places.

13.5 Mixed probability application problems

Application problems involving the normal distribution cover a wide range of topics. Such questions will not only incorporate theory associated with the normal distribution but may also include other areas of probability you have previously studied.
The amount of instant porridge oats in packets packed by a particular machine is normally distributed with a mean of $\mu$ grams and a standard deviation of 6 grams. The advertised weight of a packet is 500 grams.

- **a** Find the proportion of packets that will be underweight (less than 500 grams) when $\mu = 505$ grams.
- **b** Find the value of $\mu$ required to ensure that only 1% of packets are underweight.
- **c** As a check on the setting of the machine, a random sample of 5 boxes is chosen and the setting is changed if more than one of them is under-weight. Find the probability that the setting on the machine is changed when $\mu = 505$ grams.

## Worked Example 6

### THINK

- **a** 1 Rewrite the information in the question using appropriate notation.
  - 2 Use CAS to find $\Pr(X < 500)$.
- **b** 1 State the known probability.
  - 2 Find the corresponding standardised value, $Z$, by using CAS.
  - 3 Write the standardised formula connecting $z$ and $x$.
  - 4 Substitute the appropriate values and solve for $\mu$.
- **c** 1 The wording of the question (sample of 5 boxes) indicates that this is now a binomial distribution. Rewrite the information in the question using appropriate notation.
  - 2 Using CAS, calculate the probability.

### WRITE

- **a** $X$ is the amount of instant porridge oats in a packet and $X \sim N(505, 6^2)$.
  - $\Pr(X < 500) = 0.2023$
- **b** $\Pr(X < 500) = 0.01$
  - $\Pr(Z < z) = 0.01$
  - $z = -2.3263$
  - $z = \frac{x - \mu}{\sigma}$
  - $-2.3263 = \frac{500 - \mu}{6}$
  - $\mu = 513.96 \text{ g}$
- **c** Let $Y$ = the number of underweight packets.
  - $Y \sim Bi(5, 0.2023)$
  - $\Pr(Y > 1) = 1 - \Pr(Y \leq 1)$
  - $= 1 - 0.7325$
  - $= 0.2674$

## EXERCISE 13.5

### Mixed probability application problems

1 **WE6** Packages of butter with a stated weight of 500 grams have an actual weight of $W$ grams, which is normally distributed with a mean of 508 grams.

- **a** If the standard deviation of $W$ is 3.0 grams, find:
  - i the proportion of packages that weigh less than 500 grams
  - ii the weight that is exceeded by 99% of the packages.
b If the probability that a package weighs less than 500 grams is not to exceed 0.01, find the maximum allowable standard deviation of \( W \).

2 Chocolate Surprise is a toy that is packed inside an egg-shaped chocolate. A certain manufacturer provides four different types of Chocolate Surprise toy — a car, an aeroplane, a ring and a doll — in the proportions given in the table.

<table>
<thead>
<tr>
<th>Toy</th>
<th>Proportion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Car</td>
<td>( 3k^2 + 2k )</td>
</tr>
<tr>
<td>Aeroplane</td>
<td>( 6k^2 + 2k )</td>
</tr>
<tr>
<td>Ring</td>
<td>( k^2 + 2k )</td>
</tr>
<tr>
<td>Doll</td>
<td>( 3k )</td>
</tr>
</tbody>
</table>

a Show that \( k \) must be a solution to the equation \( 10k^2 + 9k - 1 = 0 \).

b Find the value of \( k \).

In response to customer demand, the settings on the machine that produce Chocolate Surprise have been changed so that 25% of all Chocolate Surprises produced contain rings. A sample of 8 Chocolate Surprises is randomly selected from a very large number produced by the machine.

c What is the expected number of Chocolate Surprises that contain rings? Give your answer correct to the nearest whole number.

d What is the probability, correct to 4 decimal places, that this sample has exactly 2 Chocolate Surprises that contain rings?

e What is the smallest sample size that should be taken so that the probability of selecting no Chocolate Surprise that contain a ring is at most 0.09?

A Chocolate Surprise is considered defective if it weighs less than 100 grams. The weight of a Chocolate Surprise is known to be normally distributed with a mean of 125 grams.

f If 8.2% of the Chocolate Surprises produced are defective, find, to the nearest gram, the standard deviation for the weight of the Chocolate Surprises.

3 A particular brand of car speedometer was tested for accuracy. The error measured is known to be normally distributed with a mean of 0 km/h and a standard deviation of 0.76 km/h. Speedometers are considered unacceptable if the error is more than 1.5 km/h. Find the proportion of speedometers which are unacceptable.

4 The heights of adult males in Perth can be taken as normally distributed with a mean of 174 cm and a standard deviation of 8 cm. Suppose the West Australian Police Force accepts recruits only if they are at least 180 cm tall.

a What percentage of Perth adult males satisfy the height requirement for the West Australian Police Force?

b What minimum height, to the nearest centimetre, would the West Australian Police Force have to accept if it wanted a quarter of the Perth adult male population to satisfy the height requirement?
5 a Farmer David grows avocados on a farm on Mount Tamborine, Queensland. The average weight of his avocados is known to be normally distributed with a mean weight of 410 grams and a standard deviation of 20 grams.
   i Find the probability that an avocado chosen at random weighs less than 360 grams.
   ii Find the probability that an avocado that weighs less than 360 grams weighs more than 340 grams.

b Farmer Jane grows avocados on a farm next to farmer David’s. If $Y$ represents the average weight of Jane’s avocados, the weights of which are also normally distributed where $\Pr(Y < 400) = 0.4207$ and $\Pr(Y > 415) = 0.3446$, find the mean and standard deviation of the weights of Jane’s avocados. Give answers correct to the nearest integer.

6 A manufacturer produces metal rods whose lengths are normally distributed with a mean of 145.0 cm and a standard deviation 1.4 cm.
   a Find the probability, correct to 4 decimal places, that a randomly selected metal rod is longer than 146.5 cm.
   b A metal rod has a size fault if its length is not within $d$ cm either side of the mean. The probability of a metal rod having a size fault is 0.15. Find the value of $d$, giving your answer correct to 1 decimal place.
   c A random sample of 12 metal rods is taken from a crate containing a very large number of metal rods. Find the probability that there are exactly 2 metal rods with a size fault, giving your answer correct to 4 decimal places.
   d The sales manager is considering what price, $x$ dollars, to sell each of the metal rods for, whether they are good or have some kind of fault. The materials cost is $5 per rod. The metal rods are sorted into three bins. The staff know that 15% of the manufactured rods have a size fault and another 17% have some other fault. The profit, $Y$ dollars, is a random variable whose probability distribution is shown in the following table.

<table>
<thead>
<tr>
<th>Bin</th>
<th>Description</th>
<th>Profit ($y$)</th>
<th>$\Pr(Y = y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Good metal rods that are sold for $x$ dollars each</td>
<td>$x - 5$</td>
<td>$a$</td>
</tr>
<tr>
<td>B</td>
<td>Metal rods with a size fault — these are not sold but recycled.</td>
<td>0</td>
<td>0.15</td>
</tr>
<tr>
<td>C</td>
<td>Metal rods with other faults — these are sold at a discount of $3 each.</td>
<td>$x - 8$</td>
<td>0.17</td>
</tr>
</tbody>
</table>

   i Find the value of $a$, correct to 2 decimal places.
   ii Find the mean of $Y$ in terms of $x$.
   iii Hence or otherwise, find, correct to the nearest cent, the selling price of good rods so that the mean profit is zero.
   iv The metal rods are stored in the bins until a large number is ready to be sold. What proportion of the rods ready to be sold are good rods?
7 A company sells two different products, X and Y, for $5.00 and $6.50 respectively. Regular markets exist for both products, with sales being normally distributed and averaging 2500 units (standard deviation 700) and 3000 units (standard deviation 550) respectively each week. It is company policy that if in any one week the market for a particular product falls below half the average, that product is advertised as a ‘special’ the following week.

a Find the probability, correct to 4 decimal places, that product X will be advertised as a ‘special’ next week.

b Find the probability, correct to 4 decimal places, that product Y will be advertised as a ‘special’ next week.

c Find the probability, correct to 4 decimal places, that both products will be advertised as a ‘special’ next week.

d If 40% of the company’s product is product X and 60% is product Y, find the probability that:

i one product is a ‘special’

ii if one product is advertised as ‘special’, then it is product X.

8 The height of plants sold at a garden nursery supplier are normally distributed with a mean of 18 cm and a standard deviation of 5 cm.

a Complete the following table by finding the proportions for each of the three plant sizes, correct to 4 decimal places.

<table>
<thead>
<tr>
<th>Description of plant</th>
<th>Plant size (cm)</th>
<th>Cost in $</th>
<th>Proportion of plants</th>
</tr>
</thead>
<tbody>
<tr>
<td>Small</td>
<td>Less than 10 cm</td>
<td>2.00</td>
<td></td>
</tr>
<tr>
<td>Medium</td>
<td>10–25 cm</td>
<td>3.50</td>
<td></td>
</tr>
<tr>
<td>Large</td>
<td>Greater than 25 cm</td>
<td>5.00</td>
<td></td>
</tr>
</tbody>
</table>

b Find the expected cost, to the nearest dollar, for 150 plants chosen at random from the garden nursery.

9 A fruit grower produces peaches whose weights are normally distributed with a mean of 185 grams and a standard deviation of 20 grams.

Peaches whose weights exceed 205 grams are sold to the cannery, yielding a profit of 60 cents per peach. Peaches whose weights are between 165 grams and 205 grams are sold to wholesale markets at a profit of 45 cents per peach. Peaches whose weights are less than 165 grams are sold for jam at a profit of 30 cents per peach.

a Find the percentage of peaches sold to the canneries.

b Find the percentage of peaches sold to the wholesale markets.

c Find the mean profit per peach.
10 The Lewin Tennis Ball Company makes tennis balls whose diameters are distributed normally with a mean of 70 mm and a standard deviation of 1.5 mm. The tennis balls are packed and sold in cylindrical tins that each hold five tennis balls. A tennis ball fits in the tin if the diameter is less than 71.5 mm.

a What is the probability, correct to 4 decimal places, that a randomly chosen tennis ball produced by the Lewin company fits into the tin?

The Lewin management would like each ball produced to have a diameter between 68.6 mm and 71.4 mm.

b What is the probability, correct to 4 decimal places, that a randomly chosen tennis ball produced by the Lewin company is in this range?

c A tin of five balls is selected at random. What is the probability, correct to 4 decimal places, that at least one ball has a diameter outside the range of 68.6 mm to 71.4 mm?

Lewin management wants engineers to change the manufacturing process so that 99.5% of all balls produced have a diameter between 68.6 mm and 71.4 mm. The mean is to stay at 70 mm but the standard deviation is to be changed.

d What should the new standard deviation be, correct to 4 decimal places?

11 The Apache Orchard grows a very juicy apple called the Fugee apple. Fugee apples are picked and then sorted by diameter in three categories:

- small — diameter less than 60 mm
- jumbo — the largest 15% of the apples
- standard — all other apples.

Diameters of Fugee apples are found to be normally distributed with a mean of 71 mm and a standard deviation of 12 mm.

a A particular apple is the largest possible whose diameter lies within two standard deviations of the mean. What is the diameter? Give your answer correct to the nearest millimetre.

b Find, correct to 4 decimal places, the probability that a Fugee apple, selected at random, has a diameter less than 85 mm.

c What percentage of apples (to the nearest 1 per cent) is sorted into the small category?

d Find, correct to the nearest millimetre, the minimum diameter of a jumbo Fugee.

e An apple is selected at random from a bin of jumbo apples. What is the probability, correct to 4 decimal places, that it has a diameter greater than 100 mm?

f The Apache Orchard receives the following prices for Fugee apples:

- small — 12 cents each
- standard — 15 cents each
- jumbo — 25 cents each.

What is the expected income, correct to the nearest dollar, for a container of 2500 unsorted apples?

g Some apples are selected before sorting and are packed into bags of six to be sold at the front gate of the orchard. Find the probability, correct to 4 decimal places, that one of these bags contains at least two jumbo apples.
12 A brand of disinfectant is sold in two sizes: standard and large. For each size, the contents, in litres, of a randomly chosen bottle is normally distributed with a mean and standard deviation as shown in the following table.

<table>
<thead>
<tr>
<th>Bottle size</th>
<th>Mean</th>
<th>Standard deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard</td>
<td>0.765 L</td>
<td>0.007 L</td>
</tr>
<tr>
<td>Large</td>
<td>1.015 L</td>
<td>0.009 L</td>
</tr>
</tbody>
</table>

a Find the probability, correct to 4 decimal places, that a randomly chosen standard bottle contains less than 0.75 litres.
b Find the probability that a box of 12 randomly chosen large bottles contains at least 4 bottles whose contents are each less than 1 litre.

13 Amalie is gathering data on two particular species of yellow butterflies: the lemon emigrant and the yellow emigrant, which can be very difficult to tell apart as the intensity of the yellow can be confusing. Both species are equally likely to be caught in a particular area of Australia. One technique for telling them apart is by measuring the lengths of their antennae. For the lemon emigrant, the antennae are distributed normally with a mean of 22 mm and a standard deviation of 1.5 mm.

a Find the probability, correct to 4 decimal places, that a randomly chosen lemon emigrant butterfly will have antennae which are shorter than 18 mm.
b Amalie knows that 8% of the yellow emigrants have antennae which are shorter than 15.5 mm, and 8% of yellow emigrant butterflies have antennae which are longer than 22.5 mm. Assuming that the antenna lengths are normally distributed, find the mean and standard deviation of the antenna length of yellow emigrant butterflies, giving your answers correct to the nearest 0.1 mm.

In the region where Amalie is hunting for yellow butterflies, 45% of the yellow butterflies are lemon emigrants and 55% are lemon emigrants.

c Find the probability, correct to 4 decimal places, that a random sample of 12 butterflies from the region will contain 5 yellow emigrant butterflies.

14 The daily error (in seconds) of a particular brand of clock is known to be normally distributed. Only those clocks with an error of less than 3 seconds are acceptable.

a Find the mean and standard deviation of the distribution of error if 2.5% of the clocks are rejected for losing time and 2.5% of the clocks are rejected for gaining time.
b Determine the probability that fewer than 2 clocks are rejected in a batch of 12 such clocks.
The Maths Quest Review is available in a customisable format for you to demonstrate your knowledge of this topic.

The Review contains:

- **Short-answer** questions — providing you with the opportunity to demonstrate the skills you have developed to efficiently answer questions without the use of CAS technology
- **Multiple-choice** questions — providing you with the opportunity to practise answering questions using CAS technology
- **Extended-response** questions — providing you with the opportunity to practise exam-style questions.

A summary of the key points covered in this topic is also available as a digital document.

### REVIEW QUESTIONS

Download the Review questions document from the links found in the Resources section of your eBookPLUS.

---

**Activities**

To access eBookPLUS activities, log on to [www.jacplus.com.au](http://www.jacplus.com.au)

**Interactivities**

A comprehensive set of relevant interactivities to bring difficult mathematical concepts to life can be found in the Resources section of your eBookPLUS.

studyON is an interactive and highly visual online tool that helps you to clearly identify strengths and weaknesses prior to your exams. You can then confidently target areas of greatest need, enabling you to achieve your best results.
13 Answers

EXERCISE 13.2

1 a \( \mu = 2, \sigma = 3 \)

b $f(x)$

\[
\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-2)^2}{2}} \, dx = 0.9999 \approx 1
\]

2 b \( \mu = 3, \sigma = 1 \)

3 a \(0.025\)  b \(0.0015\)  c \(0.84\)

4 0.0015

5 b \( \mu = -2 \)

6 a \( \mu = 1, \sigma = 0.3 \text{ or } \frac{3}{10} \)

b Dilation factor \( \frac{10}{3} \) parallel to the y-axis, dilation factor \( \frac{3}{10} \) parallel to the x-axis, translation 1 unit in the positive x-direction

c $f(x)$

7 a \( \mu = -4, \sigma = 10 \)

b Dilation factor \( \frac{1}{10} \) from the x-axis, dilation factor 10 from the y-axis, translation 4 units in the negative x-direction

c i \(100\)  ii \(116\)

d $\int_{-\infty}^{\infty} \frac{1}{10\sqrt{2\pi}} e^{-\frac{(x+4)^2}{10}} \, dx = 0.9999 \approx 1$}

8 a \( \mu = 2, \sigma = \frac{2}{5} \)

b \( \frac{10^4}{25} = 4.16 \)

c i \(10\)  ii \(2.8\)

9 a i \(100 \text{ and } 140\)  ii \(80 \text{ and } 160\)  iii \(60 \text{ and } 180\)

b i \(0.025\)  ii \(0.0015\)

10 0.15 %

11 a 10 and 20  b 5 and 25  c 0 and 30

12 a 0.84  b 0.815  c 0.9702

13 a 0.975  b 0.95  c 0.9744

14 a 0.815  b \( k = 205 \)  c \( h = 155 \)

15 a 0.04  b 1.04

c i \(5\)  ii \(-1.92\)

16 a 0.815  b 0.025  c 0.94

d \( m = 80.9 \)

EXERCISE 13.3

1 a i \(0.8849\)  ii \(0.7703\)

b i \(0.9088\)  ii \(-\frac{4}{3}\)

2 0.4088

3 a 0.9772  b 0.0228  c 0.9545

d 0.0512

4 a 0.0912  b 0.2119  c 0.7501

5 a 0.0228  b 0.0912  c 0.8860

6 a 2.76%  b 92.17%

7 a i \(0.65\)  ii \(0.27\)

b i \(0.29\)  ii \(0.5179\)

c i \(k = 2.4\)  ii \(k = 1.6\)

8 Jing Jing

9 The salmon is in the top 26.6%, so it is not gourmet.

10 Chemistry 0.8, Maths Methods 0.86, Physics 0.875 so Physics best

11 a 0.1587  b 0.1587  c 0.3695

12 a 6.68%  b 0.19%  c 0.88%

13 a 0.0336  b 0.5684  c 0.9762

d 0.3081

14 \( k = 25.2412 \)

EXERCISE 13.4

1 a \( a = 34.34 \)  b \( a = 14.68 \)  c \( a = 1.65 \)

2 \( m = 12.9444, n = 41.6556 \)
\[
\begin{align*}
3 & \sigma = 15.5 \\
4 & \mu = 37.6 \\
5 & a = -0.2793 \quad b = 1.0364 \quad c = 0.3585 \\
6 & a = 42.52 \quad b = 41.53 \quad c = 13.40 \\
7 & a = 0.1764 \quad b = 0.3319 \\
8 & a = 49.4443 \quad b = 36.6489 \\
9 & \mu = 13.35 \\
10 & \mu = 26 \\
11 & \sigma = 10.5 \\
12 & \text{SD}(X) = 12.2 \\
13 & \mu = 34.6, \sigma = 2.5 \\
14 & \mu = 15.8, \sigma = 5.2 \\
15 & 0.6842 \\
16 & \mu = 37.68, \sigma = 11.21
\end{align*}
\]

**EXERCISE 13.5**

1. a. i. \(0.0038\)  
   b. 3.4389 grams

2. a. \(3k^2 + 2k + 6k^2 + 2k + k^2 + 2k + 3k = 1\) 
   \(10k^2 + 9k - 1 = 0\)
   b. \(k = \frac{1}{10}\)  
   c. 2  
   d. 0.3115
   e. 9  
   f. \(\sigma = 18\)

3. 0.0484

4. a. 22.66\%  
   b. 179 cm

5. a. i. \(0.0062\)  
   b. \(\mu = 405, \sigma = 25\)
   c. 0.2924
   d. i. \(a = 0.68\)  
   ii. \(0.85x - 4.76\)  
   iii. 6 cents  
   iv. 80\%

6. a. \(0.1420\)  
   b. \(d = 2.0\)  
   c. \(\mu = 405, \sigma = 25\)
   d. i. \(a = 0.68\)  
   ii. \(0.85x - 4.76\)  
   iii. 6 cents  
   iv. 80\%

7. a. \(0.0371\)  
   b. \(0.0032\)  
   c. \(0.0001\)

8. a. Small: 0.0548, medium: 0.8644, large: 0.0808
   b. $531

9. a. 15.87\%  
   b. 68.27\%  
   c. 45 cents

10. a. \(0.8413\)  
    b. \(\mu = 405, \sigma = 25\)
    c. \(\mu = 405, \sigma = 25\)
    d. \(\sigma = 0.4987\)

11. a. 95 mm  
    b. 0.8783  
    c. 18\%  
    d. 83 mm  
    e. 0.0078  
    f. $399

12. a. \(0.0161\)  
    b. \(0.0019\)

13. a. \(0.0038\)
    b. \(\mu = 19.0 \text{ mm}, \sigma = 2.5 \text{ mm}\)
    c. 0.2225

14. a. \(\mu = 0, \sigma = 1.5306\)
    b. 0.8816